

Hopf reductions, fluxes and branes

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Abstract

We use a series of reductions, T -dualities and liftings to construct connections between fractional brane solutions in IIA, IIB and M-theory. We find a number of phantom branes that are not supported by the geometry, however materialize upon untwisting and/or Hopf-reduction.

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1. Introduction and discussion

The study of M-theory backgrounds with $U(1)$ isometries and their associated Hopf reductions dates back to the eighties (see e.g. [1,2]). In the presence of branes and depending on whether the direction of the reduction is parallel or transverse to the brane, the dimensionality of the brane worldvolume may or may not change under the reduction. A subsequent T -duality transformation can be applied. In the case of $AdS_{D-d} \times S^d$ backgrounds the fact that both the AdS_{D-d} and the S^d factors can be thought of as $U(1)$ fibrations has been used extensively in order to establish new vacua through Hopf-reductions/ T -dualities. One of the lessons was that these operations can untwist the $U(1)$ bundles and break supersymmetry at the level of supergravity solutions (although this may not be true at the level of full string theory) [3,4,5]. The recent interest in supergravity solutions involving branes wrapping cycles in conifold backgrounds motivates us to investigate some reductions and dualities in this case.

Our starting point is the singular type IIB solution of [6], (the KT solution). The geometry is of the form (warped) $\mathbb{R}^{1,3} \times \mathcal{C}(T^{1,1})$ where the transverse space $\mathcal{C}(T^{1,1})$ is the conifold of [7], which is a cone over $T^{1,1}$. The latter is topologically $S^2 \times S^3$. Since this can be thought of as a $U(1)$ fibration over $S^2 \times S^2$, we can Hopf-reduce/ T -dualize along the $U(1)$. The KT solution involves a system of ordinary and fractional D3 branes. The latter are thought of in this context as D5 branes wrapping the S^2 of $T^{1,1}$. The T -duality in this case will be along a direction transverse to the branes.

A closely related non-singular system is the KS solution [8]. In this case the transverse space is instead the deformed conifold of [7]. The latter is obtained from the conifold by replacing the apex by an S^3 . In the context of KS, the D5 wrapping the S^2 produces flux through the S^3 , which is thus stabilized (kept finite). The deformed conifold is also a $U(1)$ fibration, although Hopf-reducing is more complicated technically.

In a related development [9,10], it was realized that a system of D6 branes in IIA theory wrapping the S^3 of the resolved conifold (the AMV solution) can be lifted in M-theory to a manifold of G_2 holonomy. The resolved conifold is obtained from the conifold by replacing the apex by an S^2 . In the case of AMV, the D6-branes wrapping the S^3 produce flux through the S^2 , which is stabilized.¹ Once more, metamorphoses of $U(1)$ bundles in this and related setups provide interesting connections.

¹ In the terminology of [10], this is the situation where the D6 branes have disappeared and have been replaced by the F_2 flux. Here we use the notion of “brane” modulo the large N duality of [11], namely both for N D6-branes wrapping T^*S^3 and for the physically equivalent case of the resolved conifold with N units of flux through the S^2 .

This paper contains a series of Hopf-reductions, T -dualities and liftings involving branes in the geometries discussed above. There are two points that we find worth emphasizing:

While T -duality can untwist the $U(1)$ bundles, the M-theory lifting (in the presence of fractional branes) can induce new twistings. This has received much attention lately due to [10]. In the liftings considered here, orientation issues play a subtle role. Starting from IIB theories in a background involving a $T^{1,1}$ space we may end up in an M-theory background with a different $T^{1,1}$.

We find a number of examples with twisted geometries (with respect to the $U(1)$ fibre) where there are fluxes, but where there are no corresponding non-contractible cycles for the fluxes to be integrated over into charges. We think of these backgrounds as phantom branes that are not supported by the (twisted) geometry. However, there are two instances in which real branes can emerge from phantom ones. *a)* By switching-off some fluxes (typically corresponding to fractional branes) the geometry may get untwisted giving thus rise to the cycles necessary for supporting the branes. Note that this is not a continuous process since it involves taking the number of fractional branes (an integer) to zero. *b)* By Hopf-reduction.

In a way this could be considered the discrete analogue of the large- N duality between fluxes in twisted geometry and branes. A somewhat similar phenomenon which has already appeared in the literature is the case of winding strings in the background of the Kaluza-Klein monopole [12]. We comment on this in section 3.

The outline of the paper is as follows. Since the properties of $T^{1,1}$ spaces will be important in the following, we review these in section 2. Special attention is paid to issues related to orientations of cycles. The T -dual and the M-theory lift of the KT solution and its S -dual are discussed in sections 3 and 4 respectively. Here we find a number of phantom branes that are not supported by the geometry, however materialize upon Hopf-reduction and/or untwisting. Section 5 is more speculative and concerns fivebranes in the geometry of AMV. Finally in section 6 we explore the $U(1)$ isometry of the deformed conifold and find a new set of variables in which this isometry is explicit. Some useful formulae concerning different reductions and T -dualities are collected in the appendix.

2. The geometry of $T^{p,q}$ spaces

$T^{p,q}$ spaces can either be thought of as $U(1)$ fibrations over $S^2 \times S^2$ or as $SU(2) \times SU(2)/U(1)$ coset spaces. Let $0 \leq \phi_i \leq 2\pi$, $0 \leq \theta_i \leq \pi$, $i = 1, 2$ parametrize the two S^2 and let $0 \leq \psi \leq 4\pi$ be the coordinate of the $U(1)$ fibre. The most general $T^{p,q}$ metric reads

$$\begin{aligned} ds_T^2 = & D(d\psi + p \cos\theta_1 d\phi_1 + q \cos\theta_2 d\phi_2)^2 + A(\sin^2\theta_1 d\phi_1^2 + d\theta_1^2) \\ & + C(\sin^2\theta_2 d\phi_2^2 + d\theta_2^2) + 2B[\cos\psi(d\theta_1 d\theta_2 - \sin\theta_1 \sin\theta_2 d\phi_1 d\phi_2) \\ & + \sin\psi(\sin\theta_1 d\phi_1 d\theta_2 + \sin\theta_2 d\phi_2 d\theta_1)] \end{aligned} \quad (2.1)$$

where A, B, C, D are constants.

It is convenient to introduce the following basis of one-forms, motivated by the geometry of coset spaces [13]:

$$\begin{aligned} \begin{pmatrix} e^1 \\ e^2 \end{pmatrix} &= \begin{pmatrix} \sin\theta_1 d\phi_1 \\ d\theta_1 \end{pmatrix} \\ \begin{pmatrix} e^3 \\ e^4 \end{pmatrix} &= \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \sin\theta_2 d\phi_2 \\ d\theta_2 \end{pmatrix} \\ e^5 &= d\psi + A_1^{(p,q)}; \quad A_1^{(p,q)} = p \cos\theta_1 d\phi_1 + q \cos\theta_2 d\phi_2. \end{aligned} \quad (2.2)$$

$A_1^{(p,q)}$ is the connection 1-form of the (p, q) $U(1)$ bundle over $S^2 \times S^2$. In terms of the above base the metric (2.1) becomes

$$ds_T^2 = D(e^5)^2 + A((e^1)^2 + (e^2)^2) + C((e^3)^2 + (e^4)^2) + 2B(e^1 e^3 + e^2 e^4) \quad (2.3)$$

Of particular interest to us is the space $T^{1,1}$ which is topologically $S^3 \times S^2$. A basis for the harmonic representatives of the (one-dimensional) spaces $H^2(T^{1,1}, \mathbb{Z})$, $H^3(T^{1,1}, \mathbb{Z})$ was constructed in [14]:

$$\begin{aligned} \omega_2 &= (\sin\theta_1 d\phi_1 \wedge d\theta_1 - \sin\theta_2 d\phi_2 \wedge d\theta_2) \\ &= e^1 \wedge e^2 - e^3 \wedge e^4 \\ \omega_3 &= e^5 \wedge \omega_2; \end{aligned} \quad (2.4)$$

Both ω_2 and ω_3 are closed. Locally we can write

$$\omega_2 = -d\omega_1; \quad \omega_1 = \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2 \quad (2.5)$$

We denote by S^2 , S^3 the basis of homology cycles for the spaces $H_2(T^{1,1}, \mathbb{Z})$, $H_3(T^{1,1}, \mathbb{Z})$ so that

$$\int_{S^3} \omega_3 = \int_{S^2} \omega_2 = 1 \quad (2.6)$$

Since $b_2(T^{1,1}) = 1$ ($b_3(T^{1,1}) = 1$), there is only one S^2 (S^3) homologically. Consider the two different 3-spheres corresponding to the orbits of each of the two $SU(2)$ factors in the isometries of $T^{1,1}$. Let us denote them by $\Sigma_3^{(i)}$, $i = 1, 2$. They are parametrized by (ϕ_i, θ_i, ψ) . Similarly, let us denote by $\Sigma_2^{(i)}$ the two 2-spheres parametrized by (ϕ_i, θ_i) , $i = 1, 2$. The homology basis can be written as [14,15]:

$$S^3 = \Sigma_3^{(1)} - \Sigma_3^{(2)}; \quad S^2 = \Sigma_2^{(1)} - \Sigma_2^{(2)}. \quad (2.7)$$

Following [14,15] we will interpret a (fractional) brane wrapping $\Sigma^{(1)}$ as being equivalent to an antibrane wrapping $\Sigma^{(2)}$. Under the transformation $\theta_2 \leftrightarrow \pi - \theta_2$, ω_1 becomes the $U(1)$ connection over a $T^{1,1}$ with *reversed* orientation. We have,

$$\begin{aligned} A_1 &\leftrightarrow \omega_1 \\ \omega_2 &\leftrightarrow dA_1 \\ \Sigma_{2,3}^{(2)} &\leftrightarrow -\Sigma_{2,3}^{(1)}. \end{aligned} \quad (2.8)$$

where $A_1 := A_1^{(1,1)}$. Under this orientation-reversal, a brane wrapping $\Sigma^{(2)}$ becomes an antibrane and vice versa. We also note that (2.6) can be written as

$$\int_{\Sigma^{(1)}} \omega - \int_{\Sigma^{(2)}} \omega = 1. \quad (2.9)$$

On the other hand,

$$\int_{\Sigma^{(1)}} \omega + \int_{\Sigma^{(2)}} \omega = 0. \quad (2.10)$$

Taking (2.8) into account, a similar set of relations can be easily derived for the integrals of dA_1 , $e^5 \wedge dA_1$ over $\Sigma_{2,3}$.

Finally we can define the $T^{1,-1}$ space which differs from $T^{1,1}$ in the relative sign between the two factors in the $U(1)$ connection or, equivalently, in the relative orientation of the two S^2 in the base. An equation similar to (2.8) relates the $T^{1,-1}$ with its orientation-reversal. Note that A_1 can be either the $U(1)$ connection of a $T^{1,1}$ or of a $T^{1,-1}$ with reversed orientation. Similarly, ω_1 can be either the $U(1)$ connection of a $T^{1,-1}$ or of a $T^{1,1}$ with reversed orientation.

3. The Klebanov-Tseytlin solution.

We start from the ten-dimensional IIB singular solution for N ordinary and M fractional D3 branes, presented in [6]. The geometry is of the form (warped) $R^{1,3} \times \mathcal{C}(T^{1,1})$ where the transverse space $\mathcal{C}(T^{1,1})$ is the conifold of [7] which is a cone over $T^{1,1}$. $T^{1,1}$ has a $U(1)$ isometry and we can apply the techniques of Hopf-reduction/ T -duality explained in the appendix. Reducing along the $U(1)$ to nine-dimensional IIB, dualizing to IIA and lifting back to ten dimensions, we find that the $U(1)$ bundle gets “untwisted” so that the transverse space is foliated by $\Sigma_2^{(1)} \times \Sigma_2^{(2)} \times S^1$. We will see how in a further lifting to eleven-dimensions a new “twisted” direction develops so that the transverse space is foliated by $T^{1,1} \times S^1$.

The KT solution reads

$$\begin{aligned} e^\phi &= g_s; & B_2 &= 3g_s M \ln \frac{r}{r_0} \omega_2; & F_3 &= M \omega_3 \\ F_5 &= \mathcal{F}_5 + {}^* \mathcal{F}_5; & \mathcal{F}_5 &= (N + 3g_s M^2 \ln \frac{r}{r_0}) \omega_2 \wedge \omega_3 \end{aligned} \quad (3.1)$$

where ω_2, ω_3 where defined in the previous section. The metric is

$$ds_{10}^2 = h^{-\frac{1}{2}} dx^\mu dx_\mu + h^{\frac{1}{2}} (dr^2 + r^2 ds_T^2) \quad (3.2)$$

where

$$h(r) = b_0 + \frac{c \times g_s}{r^4} (N + \frac{3}{4} g_s M^2 + 3g_s M^2 \ln \frac{r}{r_0}) \quad (3.3)$$

The metric ds_T^2 is the $T^{1,1}$ metric defined in the previous section for the special case $D = 1/9, A = 1/6, B = 0$. With this choice of constants $T^{1,1}$ becomes Einstein. The constant b_0 can be fixed by demanding that the asymptotically flat limit of the metric be the standard Minkowski. Similarly the constant c can be fixed by requiring that for $M = 0$ the metric reduce to the standard $AdS_5 \times S^5$ in the near-horizon limit. We also have ²,

$$\int_{S^3} F_3 = M; \quad \int_{T^{1,1}} F_5 = N + 3g_s M^2 \ln \frac{r}{r_0} \quad (3.4)$$

The identification of fractional D3 branes with D5 branes wrapping the S^2 of the conifold, was justified in [16].

² We are dropping numerical constants of proportionality.

3.1. The IIA versions of KT solution

As already mentioned, reducing the KT solution along the $U(1)$ fibre to nine-dimensional IIB, dualizing to IIA and lifting to ten-dimensional IIA, we find that the $U(1)$ bundle gets “untwisted” so that the transverse space is foliated by $\Sigma_2^{(1)} \times \Sigma_2^{(2)} \times S^1$. The interpretation of the T -dualized (IIA) version of KT, depends on our choice of orientation for $\Sigma^{(2)}$. Geometrically speaking, there seems to be no reason for preferring one orientation over the other, but the brane content is entirely different in each case.

It must be noted that while reversing the orientation of a solution is also a solution to the supergravity equations, the amount of preserved supersymmetry may change. In fact, if an Einstein space that is not a round sphere has Killing spinors, its orientation reversal gives a space with no Killing spinors [17]. So trying to preserve the original amount of supersymmetry will choose an orientation for us, however we will consider both cases here. Concerning the effect of T -duality/Hopf-reduction we note that T -duality at the full string theory level always preserves as much supersymmetry as has survived the $U(1)$ compactification. The $U(1)$ compactification in the case where the compactifying direction is naturally periodic (as is here) also leaves supersymmetry unbroken at the full string theory level. At the level of supergravity however, it is known that these operations can brake the supersymmetries of p-brane solutions (see [5] for a detailed discussion).

3.1.1. T-duality

Let us start with the orientation for $\Sigma_2^{(2)}$ which is inherited from (2.7). Reducing to nine dimensions, dualizing and lifting back to ten-dimensional type IIA, we obtain the following solution

$$\begin{aligned} F_2 &= M\omega_2; & F_3 &= 3g_s M \frac{dr}{r} \wedge \omega_2 + dA_1 \wedge dz \\ F_4 &= (N + 3g_s M^2 \ln \frac{r}{r_0}) \omega_2 \wedge \omega_2 \end{aligned} \tag{3.5}$$

while the metric reads

$$ds_{10}^2 = g_s^{\frac{1}{8}} \left(\frac{1}{9} h^{\frac{1}{2}} r^2 \right)^{\frac{1}{4}} \left(h^{-\frac{1}{2}} dx^\mu dx_\mu + h^{\frac{1}{2}} (dr^2 + \frac{1}{6} r^2 \sum_{i=1}^2 (d\Sigma_2^{(i)})^2) \right) + g_s^{-\frac{7}{8}} \left(\frac{1}{9} h^{\frac{1}{2}} r^2 \right)^{-\frac{3}{4}} dz^2 \tag{3.6}$$

where $(d\Sigma_2^{(i)})^2 := \sin^2 \theta_i d\phi_i^2 + d\theta_i^2$. We see that the dualization has untwisted the $U(1)$ bundle so that the transverse space is foliated by $S^2 \times S^2 \times S^1$.

There is a charge $Q^{(6)}$ corresponding to D6 branes wrapping an $S^2 \times S^1$. This gives fractional D3 branes! Indeed, let's denote by $\Sigma_{1,2}$ the two S^2 factors. We have

$$Q^{(6)} = \int_{\Sigma_2^{(1)}} F_2 - \int_{\Sigma_2^{(2)}} F_2 = M \quad (3.7)$$

as can be seen from (2.6).

There is however zero net $Q^{(5)}$ charge corresponding to NS5 branes wrapping an S^2 ,

$$Q^{(5)} = \int_{\Sigma_2^{(1)} \times S^1} F_3 - \int_{\Sigma_2^{(2)} \times S^1} F_3 = 0 \quad (3.8)$$

Finally, there is a $Q^{(4)}$ charge corresponding to D4 branes wrapping S^1

$$Q^{(4)} = \int_{\Sigma_2^{(1)} \times \Sigma_2^{(2)}} F_4 = N + 3g_s M^2 \ln \frac{r}{r_0} \quad (3.9)$$

The brane-content is in agreement with the (naive) expectations from T -duality: The D5 (fractional D3) become D6, the D3 become D4 wrapping the T -duality circle.

3.1.2. The orientation-reversed version

We now perform an orientation-reversal transformation on the KT solution in IIA. Upon changing the orientation, the geometrical interpretation of the solution changes. This is because a brane wrapping a nontrivial cycle becomes an antibrane upon changing the orientation of the cycle. More specifically there is now zero net charge $Q^{(6)}$ corresponding to D6 branes wrapping an $S^2 \times S^1$

$$Q^{(6)} = \int_{\Sigma_2^{(1)}} F_2 + \int_{\Sigma_2^{(2)}} F_2 = 0 \quad (3.10)$$

In other words, there is an equal number of D6 branes and antibranes.

There is however nonzero net $Q^{(5)}$ charge corresponding to an NS5 brane wrapping S^2 ,

$$Q^{(5)} = \int_{\Sigma_2^{(1)} \times S^1} F_3 + \int_{\Sigma_2^{(2)} \times S^1} F_3 = 1 \quad (3.11)$$

The $Q^{(4)}$ charge corresponding to D4 branes wrapping S^1 is still given by

$$Q^{(4)} = \int_{\Sigma_2^{(1)} \times \Sigma_2^{(2)}} F_4 = N + 3g_s M^2 \ln \frac{r}{r_0} \quad (3.12)$$

The above brane content should be compared with the system of [18,19,20,15] where there are two NS5 branes, one of them stretching along the directions x^{0-5} , and the other along $x^{0-3,7,8}$. The x^6 direction is a circle. There are also two types of D4 branes along $x^{0-3,6}$. N of them going around the circle and M of them stretching between the two NS5 branes.

In order to compare to the situation at hand, we should identify $x^{4,5}$ with $\Sigma_2^{(1)}$, say, and $x^{7,8}$ with $\Sigma_2^{(2)}$. The circle x^6 should be identified with the S^1 . The constant factor N on the right-hand-side of (3.12) is coming from the D4 branes going around the circle, while the r -dependent piece should be attributed to the D4 branes stretching between the NS5 branes. Having these two types of fourbranes will be important when the solution is lifted to eleven dimensions.

3.2. The M-theory versions of KT

Lifting to eleven-dimensional supergravity will produce two distinct solutions, corresponding to the two versions of KT in type IIA. In both cases a new twisted $U(1)$ develops.

Let us start with the second case discussed in the last section (solution (3.10)-(3.12)) and lift to eleven dimensions. The solution reads

$$ds_{11}^2 = \left(\frac{1}{9}h^{\frac{1}{2}}r^2\right)^{\frac{1}{3}} \left(h^{-\frac{1}{2}}dx^\mu dx_\mu + h^{\frac{1}{2}}(dr^2 + \frac{1}{6}r^2 \sum_{i=1}^2 (d\Sigma_2^{(i)})^2) \right) + \left(\frac{1}{9}h^{\frac{1}{2}}r^2\right)^{-\frac{2}{3}} (g_s^{-1}dz^2 + g_s(d\psi - M\omega_1)^2) \quad (3.13)$$

Note that for $M = 0$ the bundle is untwisted and the transverse geometry is foliated by $S^2 \times S^2 \times S^1 \times S^1$. However in the presence of fractional branes $M \neq 0$ and a twist is induced. The transverse space becomes foliated by $T^{1,1} \times S^1$. More accurately, this is a $T^{1,1}$ space whose S^3 fibre (when $T^{1,1}$ is viewed as S^3 over S^2) is replaced by the lens space S^3/Z_M . Indeed, under the redefinition $\psi \rightarrow -M\psi$, $d\psi - M\omega_1$ becomes proportional to the covariant displacement on the $U(1)/Z_M$ fibre of a $T^{1,1}$ with reversed orientation. This is reminiscent of the situation in [5] (see also [21]).

We also have

$$F_4 = (N + 3g_s M^2 \ln \frac{r}{r_0}) \omega_2 \wedge \omega_2 - (3g_s M^2 \frac{dr}{r} \wedge \omega_2 + dA_1 \wedge dz) \wedge (d\psi - M\omega_1) \quad (3.14)$$

Note that $dF_4 = 0$ as it should. To prove that, use the fact that $\omega_2 \wedge dA_1 = 0$.

The solution contains M5-brane charge wrapping the S^2 of $T^{1,1}$,

$$Q^{(5,1)} = \int_{S^3 \times S^1} F_4 \quad (3.15)$$

This will reduce to NS5 wrapping S^2 as in (3.11).

Naively, the solution contains M5 branes wrapping both the “untwisted” S^1 and the $U(1)$ fibre of the base $\Sigma_2^{(1)} \times \Sigma_2^{(2)}$ over the $T^{1,1}$. Their charge $\tilde{Q}^{(5,2)}$ is given by

$$\tilde{Q}^{(5,2)} = \int_{\Sigma_2^{(1)} \times \Sigma_2^{(2)}} F_4 = N + 3g_s M^2 \ln \frac{r}{r_0} \quad (3.16)$$

Since there is no nontrivial $\Sigma_2^{(1)} \times \Sigma_2^{(2)}$ cycle in $T^{1,1}$ such a $\tilde{Q}^{(5,2)}$ charge can be taken to zero, even though there is non-vanishing F_4 flux. However upon reduction to IIA, these phantom M5 branes come to existence in the form of the D4 branes wrapping the S^1 , as in (3.12)! Also, as noted below (3.13), in the absence of fractional branes the $U(1)$ bundle is untwisted and again the phantom M5 branes become ordinary M5 branes wrapping $S^1 \times S^1$. In the following, all the phantom charges will carry tildes.

This is the first instance in which we encounter a phenomenon whereby there are fluxes unsupported by the geometry, which however materialize into physical branes upon untwisting and/or Hopf reduction. The $U(1)$ (un)twisting is the key to this, and in a way provides a discrete analogue of the large N duality between the fluxes and branes. We will meet more examples of this in the sequel.

Let us compare to the situation in [12] where a string with nonzero winding number in the background of a Kaluza-Klein monopole was considered. The string can unwind since the total space has $\pi_1 = 0$, but the charge associated to the winding number of the string is conserved. From this one concludes that there is a zero mode among the collective coordinates of the KK monopole which couples to the charge associated with the winding of the string. In our case $\pi_1 = Z_M$ is nonzero, but there are still no nontrivial one-cycles in homology ($b_1 = 0$).

The lifting of the solution (3.7)-(3.9) is similar. The resulting geometry is now $T^{1,-1} \times S^1$ (for $M \neq 0$) and the charge $Q^{(5,1)} = 0$. As before the S^3 fibre of the $T^{1,-1}$ is replaced by the lens space S^3/Z_M . Again, there is a flux through the contractible $\Sigma_2^{(1)} \times \Sigma_2^{(2)}$. Upon reduction this becomes the D4-brane flux.

4. The S-dual KT

We saw in the previous section that upon T -dualizing the KT solution we get a configuration with “untwisted” $\Sigma_2^{(1)} \times \Sigma_2^{(2)} \times S^1$ geometry for the level surfaces. It would be interesting to have a situation where by T -dualizing a IIB solution we get level surfaces

with twisted geometry. The reason why this does not work with the KT solution (3.1) is clear. From the T -duality transformations (A.22) we see that in order to get a twisted $U(1)$ fibration upon going from IIB to IIA, the original IIB solution would have to have $A_1^{NS} \neq 0$. Upon T -dualizing this becomes the connection $A_1^{(3)}$ of the $U(1)$ fibration in IIA. Since $H_3 = dA_2^{NS} - dA_1^{NS} \wedge dz$, this means that the original IIB solution would have to have H_3 with nonzero flux through the Hopf fibre. We can indeed obtain a solution of IIB with the aforementioned property by performing an S -duality transformation on the KT solution:

$$\begin{aligned} \tau &\rightarrow \frac{a\tau + b}{c\tau + d} \\ \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} &\rightarrow \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H_3 \\ F_3 \end{pmatrix} \end{aligned} \quad (4.1)$$

with the rest of the fields inert and

$$\tau := \chi + ie^{-\phi}; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (4.2)$$

The KT solution (3.1) transforms to

$$\begin{aligned} e^\phi &= g_s d^2 + \frac{c^2}{g_s} := \tilde{g}_s \\ \chi &= \frac{ac + bdg_s^2}{c^2 + d^2g_s^2} \\ H_3 &= 3d g_s M \frac{dr}{r} \wedge \omega_2 + c M \omega_3 \\ F_3 &= 3b g_s M \frac{dr}{r} \wedge \omega_2 + a M \omega_3 \end{aligned} \quad (4.3)$$

The metric and the five-form field strength are still given by (3.1)-(3.3). Reducing to nine dimensions T -dualizing and lifting to IIA we get two possible situations depending on the choice of relative orientation of the two S^2 . Namely, a geometry with level surfaces given either by $T^{1,1}$ with reversed orientation or by $T^{1,-1}$.

$$\begin{aligned} ds_{10}^2 &= \tilde{g}_s^{\frac{1}{8}} \left(\frac{1}{9} h^{\frac{1}{2}} r^2 \right)^{\frac{1}{4}} \left(h^{-\frac{1}{2}} dx^\mu dx_\mu + h^{\frac{1}{2}} (dr^2 + \frac{1}{6} r^2 \sum_{i=1}^2 (d\Sigma_2^{(i)})^2) \right) \\ &\quad + \tilde{g}_s^{-\frac{7}{8}} \left(\frac{1}{9} h^{\frac{1}{2}} r^2 \right)^{-\frac{3}{4}} (dz - c M \omega_1)^2 \end{aligned} \quad (4.4)$$

Note that $dz - c M \omega_1$ is proportional to the covariant displacement on the $U(1)/Z_{cM}$ fibre either of a $T^{1,1}$ with *reversed* orientation or of a $T^{1,-1}$. We have

$$\begin{aligned} F_2 &= (a - \chi c) M \omega_2 \\ F_3 &= 3d g_s M \frac{dr}{r} \wedge \omega_2 + dA_1 \wedge (dz - c M \omega_1) \\ F_4 &= (N + 3g_s M^2 \ln \frac{r}{r_0}) \omega_2 \wedge \omega_2 + 3(b - \chi d) M g_s \frac{dr}{r} \wedge \omega_2 \wedge (dz - c M \omega_1) \end{aligned} \quad (4.5)$$

The brane content is different in each case.

First we discuss the $T^{1,1}$ with reversed orientation. The solution has zero units of D6 branes wrapping S^3 ,

$$Q^{(6)} = \int_{S^2} F_2 = 0 \quad (4.6)$$

There is however nonzero net $Q^{(5)}$ charge corresponding to NS5 branes wrapping S^2 ,

$$Q^{(5)} = \int_{S^3} F_3 = -cM \quad (4.7)$$

Finally, there is a $\tilde{Q}^{(4)}$ phantom charge corresponding to D4 branes wrapping the Hopf fibre,

$$\tilde{Q}^{(4)} = \int_{\Sigma_2^{(1)} \times \Sigma_2^{(2)}} F_4 = N + 3g_s M^2 \ln \frac{r}{r_0} \quad (4.8)$$

Again, as in (3.16), note that this makes sense either upon Hopf reducing or in the absence of fractional branes in which case the $U(1)$ fibre gets untwisted.

Turning to the $T^{1,-1}$ case we find $Q^{(6)} = (a - \chi c) M$ and $Q^{(5)} = 0$. What was said before about (4.8) is true here as well.

4.1. The M-theory lift

Upon lifting to M-theory, yet another twisted $U(1)$ fibre develops. The resulting transverse geometry is foliated by a nontrivial S^1/Z_{aM} bundle over $T^{1,1}/Z_{cM}$ such that reducing the fibre along any one of the two circles produces a $T^{1,1}$ space (or a $T^{1,-1}$, by a reasoning that should be familiar by now). In this sense there is a similarity with the AMV solution in the next section.³ The transverse geometry in that case is an S^3 over S^3 bundle. Viewing the base as a Hopf fibration over S^2 and reducing along the $U(1)$ fibre

³ In spite of the similarity it is easy to see that the resulting geometry cannot be that of AMV simply because the original ten-dimensional one is not that of the resolved conifold.

produces an S^3 (up to modding out by a discrete group) over S^2 bundle, which is actually a $T^{1,1}$ space.

Analytically, the eleven-dimensional metric reads

$$ds_{11}^2 = \left(\frac{1}{9}h^{\frac{1}{2}}r^2\right)^{\frac{1}{3}} \left(h^{-\frac{1}{2}}dx^\mu dx_\mu + h^{\frac{1}{2}}(dr^2 + \frac{1}{6}r^2 \sum_{i=1}^2 (d\Sigma_2^{(i)})^2) \right) \\ + \left(\frac{1}{9}h^{\frac{1}{2}}r^2\right)^{-\frac{2}{3}} \left(\tilde{g}_s^{-1}[(dw - a M\omega_1) - \chi(dz - c M\omega_1)]^2 + \tilde{g}_s(dz - c M\omega_1)^2 \right) \quad (4.9)$$

while the four-form is given by

$$\hat{F}_4 = F_4 - F_3 \wedge [(dw - a M\omega_1) - \chi(dz - c M\omega_1)] \quad (4.10)$$

with F_4, F_3 as in (4.5).

The solution contains phantom M5 branes wrapping the two circles. Their charge $\tilde{Q}^{(5,1)}$ is given by

$$\tilde{Q}^{(5,1)} = \int_{\Sigma_2^{(1)} \times \Sigma_2^{(2)}} F_4 = N + 3g_s M^2 \ln \frac{r}{r_0} \quad (4.11)$$

Upon reduction to IIA these become the D4 branes wrapping the Hopf fibre of $T^{1,1}$, as in (4.8). Depending on the orientation there can be phantom M5-brane charge wrapping S^2 ,

$$\tilde{Q}^{(5,2)} = \int_{S^1 \times S^3} F_4 \quad (4.12)$$

which reduces to NS5 wrapping S^2 as in (4.7).

If the opposite orientation is chosen, $Q^{(5,2)} = 0$ which of course reduces to $Q^{(5)} = 0$ in IIA. The information about the previously nontrivial $\tilde{Q}^{(5,2)}$ “charge” is now carried by $Z_{aM} \times Z_{cM}$.

5. Fivebranes in the AMV Solution

The AMV solution [10] involving M-theory on a manifold of G_2 holonomy (locally an $S^3 \times \mathbb{R}^4$) reduced to ten dimensions is given by warped 4d Minkowski times the resolved conifold of [7,22]. In addition, there is a nonzero F_2 flux through the S^2 . As before, modulo the large N duality (S^3 flop), we think of this as the supergravity solution corresponding to D6 wrapping the S^3 with a stabilizing flux through S^2 and thus producing the geometry of the resolved conifold. For some related work see also [23,24,25,26,27,28,29].

From (A.22) we immediately see that upon T -dualizing this solution to IIB, we get “untwisted” $\Sigma_2^{(1)} \times \Sigma_2^{(2)} \times S^1$ geometry: In order to get a twisted $U(1)$ fibration upon going from IIA to IIB, the original IIA solution would have to have $A_1^{(2)} \neq 0$. Upon T -dualizing this becomes the connection A_1 of the $U(1)$ fibration in IIB. Since, as we see from (A.14), $\hat{F}_3 = F_3^{(1)} - dA_1^{(2)} \wedge e^5$, this means that the original IIA solution would have to have an F_3 with nonzero flux through the Hopf fibre. This would signal the presence of NS5 branes wrapping the S^2 , which are absent from the solution of [10].

This brings us to the question of whether such a configuration of NS5’s and D6’s is known and if so, what would it lift to in M-theory. Note that the fivebrane and sixbranes stabilize the S^2 and S^3 factors respectively, The S^2 which is wrapped by the NS5 upon lifting becomes the base of a twisted $U(1)$ fibration and thus is no longer able to support an M5. This is in agreement with the fact that G_2 holonomy manifolds do not have calibrated two-dimensional submanifolds.

We should therefore look for an eleven-dimensional G_4 that has a flux through the (eleven-dimensional) Hopf fibre. Reducing this will give an F_3 flux, which upon integration over the S^3 will give NS5 charge. Since there are two S^3 ’s in the original geometry, an obvious possibility is to consider a flux

$$\int_{S^3 \times S^1} G_4 \tag{5.1}$$

which upon reduction will give an NS5 wrapping S^2 . Although the ten-dimensional geometry is foliated by $T^{1,1}$, it cannot be that of the resolved conifold, since the flux of the NS three-form will stabilize the S^3 part as well. At the moment we are not able to write down explicitly such a solution containing simultaneously D6 branes wrapping S^3 and NS5 branes wrapping S^2 and having four common non-compact directions.

M-theory solutions with a flux as in (5.1) have recently appeared in the literature [26,28]. It would clearly be interesting to obtain their IIA reductions and their IIB T -duals. The hope is to provide an explicit realization of the mirror symmetry between D5 branes on S^2 of the deformed conifold in type IIB and D6 on S^3 of the resolved in type IIA, as a single T -duality along the Hopf fibre of the $T^{1,1}$.⁴ This task is complicated, however, by the form of the deformed conifold metric, as will be discussed in the next section.

⁴ Note that as we have already remarked, the existence of NS5 branes is required in addition to the D branes for this duality to work.

6. The KS Solution

Although we have already excluded the possibility that the KS solution [8] is the T -dual of the AMV solution –since T -dualizing the latter gives untwisted transverse geometry whereas the former has a twisted one– we can still try to perform the analysis of the previous sections to this case. Technically this is complicated by the fact that the metric of the deformed conifold is written in terms of variables in which the $U(1)$ isometry is implicit [13]. Here we make partial progress by identifying a set of new variables in which the $U(1)$ isometry of the metric will be manifest, but we are yet unable to write down explicitly the metric in terms of these new variables.

The KS solution reads

$$\begin{aligned} ds_{10}^2 &= h^{-1/2}(\tau) dx_\mu dx^\mu + h^{1/2}(\tau) ds_6^2 \\ F_3 &= M(e^5 \wedge g^3 \wedge g^4 + d[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)]) \\ B_2 &= g_s M[f(\tau)g^1 \wedge g^2 + k(\tau)g^3 \wedge g^4] \\ F_5 &= g_s M^2 l(\tau) g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge e^5, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} ds_6^2 &= \frac{1}{2} \epsilon^{4/3} K(\tau) \left(\frac{1}{3K(\tau)} (d\tau^2 + (e^5)^2) + \cosh^2\left(\frac{\tau}{2}\right) [(g^3)^2 + (g^4)^2] \right. \\ &\quad \left. + \sinh^2\left(\frac{\tau}{2}\right) [(g^1)^2 + (g^2)^2] \right) \end{aligned} \tag{6.2}$$

and τ is a radial coordinate. The explicit form of the functions F, f, k, l, K will not be important in the following. The one-forms g^i are defined as

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}; & g^2 &= \frac{e^2 - e^4}{\sqrt{2}} \\ g^3 &= \frac{e^1 + e^3}{\sqrt{2}}; & g^4 &= \frac{e^2 + e^4}{\sqrt{2}} \\ g^5 &= e^5 \end{aligned} \tag{6.3}$$

so that the level surfaces of (6.2) are of the general form (2.3).

As we noted in section 4, for this solution to T -dualize to a twisted transverse geometry, B_2 would have to have a nonzero component along the direction corresponding to the $U(1)$ isometry.

Trying to Hopf-reduce runs into trouble because the solution is written in terms of coordinates in which the $U(1)$ isometry is not simply a shift in ψ . In fact from (2.2), (2.3)

we see that the isometry reads

$$\begin{aligned} \psi &\rightarrow \psi + c; \\ \begin{pmatrix} \sin\theta_2 d\phi_2 \\ d\theta_2 \end{pmatrix} &\rightarrow \begin{pmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{pmatrix} \begin{pmatrix} \sin\theta_2 d\phi_2 \\ d\theta_2 \end{pmatrix} \end{aligned} \quad (6.4)$$

We would like to find a coordinate transformation for θ_2, ϕ_2 which has the effect of the second line in (6.4). Clearly this would have to be a special case of the $SU(2)$ group of isometries of the sphere

$$z \rightarrow \frac{az + b}{-\bar{b}z + \bar{a}}; \quad z := e^{i\phi_2} \tan \frac{\theta_2}{2}; \quad |a|^2 + |b|^2 = 1. \quad (6.5)$$

Indeed for $a = \cos \frac{\epsilon}{2}$, $b = \sin \frac{\epsilon}{2}$, (6.5) implies (6.4) to order $\mathcal{O}(\epsilon)$, provided we identify

$$c = \epsilon \frac{\sin \phi_2}{\sin \theta_2} \quad (6.6)$$

Explicitly the coordinate transformation reads, to $\mathcal{O}(\epsilon)$ order,

$$\begin{aligned} \psi &\rightarrow \psi + \epsilon \frac{\sin \phi_2}{\sin \theta_2} \\ \phi &\rightarrow \phi - \epsilon \sin \phi_2 \cot \theta_2 \\ \theta_2 &\rightarrow \theta_2 + \epsilon \cos \phi_2 \end{aligned} \quad (6.7)$$

It is easy to check that (6.7) leaves e^5 invariant. We have therefore succeeded in constructing the killing vector k corresponding to the $U(1)$ isometry,

$$k = \frac{\sin \phi_2}{\sin \theta_2} \frac{\partial}{\partial \psi} - \sin \phi_2 \cot \theta_2 \frac{\partial}{\partial \phi_2} + \cos \phi_2 \frac{\partial}{\partial \theta_2} \quad (6.8)$$

It remains to find a set of new variables $\psi', \theta'_2, \phi'_2$ in which k locally takes the form $k = \partial/\partial \psi'$. Rewriting (6.2) in terms of these new variables would result in a metric whose components do not depend on ψ' and Hopf reduction would proceed as before. The variables we are looking for would therefore have to satisfy the equation

$$\begin{pmatrix} \frac{\partial \theta'_2}{\partial \theta_2} & \frac{\partial \theta'_2}{\partial \phi_2} & \frac{\partial \theta'_2}{\partial \psi} \\ \frac{\partial \phi'_2}{\partial \theta_2} & \frac{\partial \phi'_2}{\partial \phi_2} & \frac{\partial \phi'_2}{\partial \psi} \\ \frac{\partial \psi'}{\partial \theta_2} & \frac{\partial \psi'}{\partial \phi_2} & \frac{\partial \psi'}{\partial \psi} \end{pmatrix} \begin{pmatrix} k^{\theta_2} \\ k^{\phi_2} \\ k^\psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (6.9)$$

One can easily verify that

$$\begin{aligned} \hat{L} \frac{1}{2} (f_1 + f_2) &= \frac{\sin \phi_2}{\sin \theta_2}; & \hat{L} \frac{1}{2} (f_1 - f_2) &= 1 \\ \hat{L} h(\sin \phi_2 \sin \theta_2) &= 0 \end{aligned} \quad (6.10)$$

where

$$\begin{aligned}\hat{L} &:= k - k^\psi \frac{\partial}{\partial \psi}; \\ f_{1,2}(\phi_2, \theta_2) &:= \arctan \left(\frac{\sin \phi_2 \pm \sin \theta_2}{\cos \phi_2 \cos \theta_2} \right)\end{aligned}\tag{6.11}$$

and h is an arbitrary function of $\sin \phi_2 \sin \theta_2$.

Taking (6.10) into account, we can write down an explicit special solution to (6.9):⁵

$$\begin{aligned}\psi' &= \psi + f_1(\phi_2, \theta_2) \\ \phi' &= \psi + \frac{1}{2} (f_1(\phi_2, \theta_2) + f_2(\phi_2, \theta_2)) \\ \theta' &= h(\sin \phi_2 \sin \theta_2)\end{aligned}\tag{6.12}$$

The Jacobian of the above transformation reads

$$J = \sin \theta_2 h'(x)|_{x=\sin \phi_2 \sin \theta_2}\tag{6.13}$$

Finally one can derive the relation of the differentials of the old coordinates in terms of the new,

$$\begin{pmatrix} d\phi_2 \\ d\theta_2 \\ d\psi \end{pmatrix} = \begin{pmatrix} -\cot \theta_2 \sin \phi_2 & \cot \theta_2 \sin \phi_2 & \frac{\cos \phi_2}{h'(\cos^2 \phi_2 + \cos^2 \theta_2 \sin^2 \phi_2) \sin \theta_2} \\ \cos \phi_2 & -\cos \phi_2 & \frac{\cos \theta_2 \sin \phi_2}{h'(\cos^2 \phi_2 + \cos^2 \theta_2 \sin^2 \phi_2)} \\ \frac{\sin \phi_2}{\sin \theta_2} & 1 - \frac{\sin \phi_2}{\sin \theta_2} & -\frac{\cos \phi_2 \cot \theta_2}{h'(\cos^2 \phi_2 + \cos^2 \theta_2 \sin^2 \phi_2)} \end{pmatrix} \begin{pmatrix} d\psi' \\ d\phi_2' \\ d\theta_2' \end{pmatrix}\tag{6.14}$$

Using the above relation, one can examine whether or not B_2 in (6.1) has a non-trivial component along the $U(1)$ fibre. The answer is that it does and therefore, as already explained, the T -dual version of KS has twisted transverse geometry.

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Note added: After this paper was posted on the hep-th archive, we were informed by I. Klebanov, A. Tseytlin and L. Pando Zayas of their related independent work which overlaps with section 3 of this paper. See the talk by I. Klebanov at the Avatars of M-theory conference (UCSB) http://online.itp.ucsb.edu/online/mtheory_c01/klebanov/.

⁵ This is not the most general form. One can add arbitrary functions of $\sin \phi_2 \sin \theta_2$ to the rhs. Also one could take arbitrary linear combinations of θ_2' , ϕ_2' in the second and third lines.

Appendix A. Review of KK on a circle.

This is a collection of useful formulae. There is nothing here that cannot be found in the literature. However we include this appendix as a self-contained set of conventions.

A.1. Reduction of 11d supergravity to 10d type IIA

The starting point is the eleven-dimensional Lagrangian

$$\hat{e}^{-1}L = \hat{R} - \frac{1}{48}\hat{F}_4^2 + C.S. \quad (\text{A.1})$$

where

$$\hat{F}_4 = d\hat{A}_3 \quad (\text{A.2})$$

Note that all eleven- (ten-) dimensional field strengths and potentials are denoted with (without) a hat. Assuming the geometry has a $U(1)$ isometry, we can put the metric in the form:

$$ds_{11}^2 = e^{-\frac{1}{6}\varphi} ds_{10}^2 + e^{\frac{4}{3}\varphi} (A_1 + dz)^2 \quad (\text{A.3})$$

where φ , A_1 , \hat{A}_3 and the metric in ds_{10}^2 , are all assumed independent of the $U(1)$ coordinate z . We also reduce the 3-form potential

$$\hat{A}_3 = A_3 + A_2 \wedge dz \quad (\text{A.4})$$

The eleven-dimensional Lagrangian becomes

$$e^{-1}L = R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{4}e^{\frac{3}{2}\varphi}F_2^2 - \frac{1}{12}e^{-\varphi}F_3^2 - \frac{1}{48}e^{\frac{1}{2}\varphi}F_4^2 + C.S. \quad (\text{A.5})$$

where:

$$F_4 = dA_3 + dA_2 \wedge A_1; \quad F_3 = -dA_2; \quad F_2 = dA_1 \quad (\text{A.6})$$

so that

$$\hat{F}_4 = F_4 + F_3 \wedge (dz + A_1) \quad (\text{A.7})$$

A.2. Reduction of 10d type IIA to 9d $N=2$

We start with the Lagrangian

$$\hat{e}^{-1}L = \hat{R} - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{4}e^{\frac{3}{2}\hat{\phi}}\hat{F}_2^2 - \frac{1}{12}e^{-\hat{\phi}}\hat{F}_3^2 - \frac{1}{48}e^{\frac{1}{2}\hat{\phi}}\hat{F}_4^2 + C.S. \quad (A.8)$$

where

$$\hat{F}_4 = d\hat{A}_3 + d\hat{A}_2 \wedge \hat{A}_1; \quad \hat{F}_3 = d\hat{A}_2; \quad \hat{F}_2 = d\hat{A}_1 \quad (A.9)$$

We assume that there is a $U(1)$ isometry so that the ten-dimensional metric can be cast in the form

$$ds_{10}^2 = e^{-\frac{1}{2\sqrt{7}}\varphi} ds_9^2 + e^{\frac{\sqrt{7}}{2}\varphi} (A_1^{(3)} + dz)^2 \quad (A.10)$$

where φ , $A_1^{(3)}$ and the nine-dimensional metric ds_9^2 do not depend on the $U(1)$ coordinate z . We reduce the potentials as follows

$$\hat{A}_1 = A_1^{(1)} + A_0 dz; \quad \hat{A}_2 = A_2^{(1)} + A_1^{(2)} \wedge dz; \quad \hat{A}_3 = A_3 + A_2^{(2)} \wedge dz \quad (A.11)$$

The Lagrangian reduces to

$$\begin{aligned} e^{-1}L = & R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{\frac{3}{2}\phi - \frac{\sqrt{7}}{2}\varphi}(F_1)^2 \\ & - \frac{1}{4}e^{-\phi - \frac{3}{\sqrt{7}}\varphi}(F_2^{(2)})^2 - \frac{1}{4}e^{\frac{3}{2}\phi + \frac{1}{2\sqrt{7}}\varphi}(F_2^{(1)})^2 - \frac{1}{4}e^{\frac{4}{\sqrt{7}}\varphi}(F_2^{(3)})^2 \\ & - \frac{1}{12}e^{-\phi + \frac{1}{\sqrt{7}}\varphi}(F_3^{(1)})^2 - \frac{1}{12}e^{\frac{1}{2}\phi - \frac{5}{2\sqrt{7}}\varphi}(F_3^{(2)})^2 - \frac{1}{48}e^{\frac{1}{2}\phi + \frac{3}{2\sqrt{7}}\varphi}(F_4)^2 + C.S. \end{aligned} \quad (A.12)$$

where

$$\begin{aligned} \phi &= \hat{\phi}; \quad F_1 = -dA_0 \\ F_2^{(1)} &= dA_1^{(1)} + dA_0 \wedge A_1^{(3)}; \quad F_2^{(2)} = -dA_1^{(2)}; \quad F_2^{(3)} = dA_1^{(3)} \\ F_3^{(1)} &= dA_2^{(1)} + dA_1^{(2)} \wedge A_1^{(3)}; \quad F_3^{(2)} = -dA_2^{(2)} + A_1^{(1)} \wedge dA_1^{(2)} + A_0 dA_2^{(1)} \\ F_4 &= dA_3 + dA_2^{(1)} \wedge A_1^{(1)} + dA_2^{(2)} \wedge A_1^{(3)} - dA_1^{(2)} \wedge A_1^{(1)} \wedge A_1^{(3)} - A_0 dA_2^{(1)} \wedge A_1^{(3)} \end{aligned} \quad (A.13)$$

so that

$$\begin{aligned} \hat{F}_4 &= F_4 + F_3^{(2)} \wedge (dz + A_1^{(3)}) \\ \hat{F}_3 &= F_3^{(1)} + F_2^{(2)} \wedge (dz + A_1^{(3)}) \\ \hat{F}_2 &= F_2^{(1)} + F_1 \wedge (dz + A_1^{(3)}) \end{aligned} \quad (A.14)$$

A.3. Reduction of 10d type IIB to 9d N=2.

We start with the Lagrangian

$$\hat{e}^{-1}L = \hat{R} - \frac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2}e^{2\hat{\phi}}(\partial\hat{\chi})^2 - \frac{1}{12}e^{\hat{\phi}}(\hat{F}_3)^2 - \frac{1}{12}e^{-\hat{\phi}}(\hat{H}_3)^2 - \frac{1}{4 \times 5!}(\hat{F}_5)^2 + C.S. \quad (\text{A.15})$$

where

$$\begin{aligned} \hat{F}_5 &= d\hat{A}_4^R - \frac{1}{2}d\hat{A}_2^{NS} \wedge \hat{A}_2^R + \frac{1}{2}d\hat{A}_2^R \wedge \hat{A}_2^{NS} = *\hat{F}_5 \\ \hat{F}_3 &= d\hat{A}_2^R - \hat{\chi}d\hat{A}_2^{NS}; \quad \hat{H}_3 = d\hat{A}_2^{NS} \end{aligned} \quad (\text{A.16})$$

We assume that there is a $U(1)$ isometry so that the ten-dimensional metric can be cast in the form

$$ds_{10}^2 = e^{-\frac{1}{2\sqrt{7}}\varphi} ds_9^2 + e^{\frac{\sqrt{7}}{2}\varphi} (A_1 + dz)^2 \quad (\text{A.17})$$

where φ , A_1 and the nine-dimensional metric ds_9^2 do not depend on the $U(1)$ coordinate z . We reduce the potentials as follows

$$\begin{aligned} \hat{A}_2^R &= A_2^R + A_1^R \wedge dz; \quad \hat{A}_2^{NS} = A_2^{NS} + A_1^{NS} \wedge dz \\ \hat{A}_4^R &= A_4^R + A_3^R \wedge dz \end{aligned} \quad (\text{A.18})$$

The Lagrangian reduces to

$$\begin{aligned} e^{-1}L &= R - \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial\chi)^2 \\ &\quad - \frac{1}{4}e^{-\phi-\frac{3}{\sqrt{7}}\varphi}(F_2^{NS})^2 - \frac{1}{4}e^{\phi-\frac{3}{\sqrt{7}}\varphi}(F_2^R)^2 - \frac{1}{4}e^{\frac{4}{\sqrt{7}}\varphi}(F_2)^2 \\ &\quad - \frac{1}{12}e^{-\phi+\frac{1}{\sqrt{7}}\varphi}(F_3^{NS})^2 - \frac{1}{12}e^{\phi+\frac{1}{\sqrt{7}}\varphi}(F_3^R)^2 - \frac{1}{48}e^{-\frac{2}{\sqrt{7}}\varphi}(F_4^R)^2 + C.S. \end{aligned} \quad (\text{A.19})$$

where

$$\begin{aligned} \phi &= \hat{\phi}; \quad \chi = \hat{\chi} \\ F_2^R &= -dA_1^R + \chi dA_1^{NS}; \quad F_2^{NS} = -dA_1^{NS}; \quad F_2 = dA_1 \\ F_3^{NS} &= dA_2^{NS} + A_1 \wedge dA_1^{NS}; \quad F_3^R = dA_2^R + dA_1^R \wedge A_1 - \chi(dA_2^{NS} + dA_1^{NS} \wedge A_1) \\ F_4^R &= -dA_3^R + \frac{1}{2}(-dA_2^{NS} \wedge A_1^R - A_2^{NS} \wedge dA_1^R + dA_2^R \wedge A_1^{NS} + A_2^R \wedge dA_1^{NS}) \end{aligned} \quad (\text{A.20})$$

so that

$$\hat{F}_n = F_n + F_{n-1} \wedge (dz + A_1) \quad (\text{A.21})$$

A.4. 9d T-duality

The two nine-dimensional Lagrangians described above, are in fact related to each other by the following local field transformations:

$$\begin{aligned}
\phi_A &\leftrightarrow \frac{3}{4}\phi_B - \frac{\sqrt{7}}{4}\varphi_B; & \varphi_A &\leftrightarrow -\frac{\sqrt{7}}{4}\phi_B - \frac{3}{4}\varphi_B; & A_0 &\leftrightarrow -\chi \\
A_1^{(1)} &\leftrightarrow -A_1^R + \chi A_1^{NS}; & A_1^{(2)} &\leftrightarrow -A_1; & A_1^{(3)} &\leftrightarrow -A_1^{NS} \\
A_2^{(1)} &\leftrightarrow A_2^{NS} - A_1^{NS} \wedge A_1; & A_2^{(2)} &\leftrightarrow -A_2^R + A_1^R \wedge A_1 \\
A_3 &\leftrightarrow -A_3^R + \frac{1}{2}A_1^{NS} \wedge A_2^R - \frac{1}{2}A_1^R \wedge A_2^{NS} - A_1^{NS} \wedge A_1^R \wedge A_1
\end{aligned} \tag{A.22}$$

Or, in terms of field strenghts:

$$\begin{aligned}
F_1 &\leftrightarrow d\chi \\
F_2^{(1)} &\leftrightarrow F_2^R; & F_2^{(2)} &\leftrightarrow F_2; & F_2^{(3)} &\leftrightarrow F_2^{NS} \\
F_3^{(1)} &\leftrightarrow F_3^{NS}; & F_3^{(2)} &\leftrightarrow F_3^R \\
F_4 &\leftrightarrow F_4^R;
\end{aligned} \tag{A.23}$$

A.5. String vs Einstein frame.

In ten dimensions the Einstein metric is related to the string metric through

$$g_{\mu\nu} = e^{-\frac{1}{2}\hat{\phi}} g_{\mu\nu}^{str} \tag{A.24}$$

The reduction of the ten-dimensional metric in the string frame reads

$$ds_{10}^{str} = e^{\frac{1}{2}(\phi - \frac{1}{\sqrt{7}}\varphi)} ds_9^2 + e^{\frac{1}{2}(\phi + \sqrt{7}\varphi)} (dz + A_1)^2 \tag{A.25}$$

In the string frame the ten-dimensional type IIB Lagrangian becomes

$$\hat{e}^{-1}L = e^{-2\hat{\phi}}(\hat{R} + 4(\partial\hat{\phi})^2 - \frac{1}{12}(\hat{H}_3)^2) - \frac{1}{2}(\partial\hat{\chi})^2 - \frac{1}{12}(\hat{F}_3)^2 - \frac{1}{4 \times 5!}(\hat{F}_5)^2 + C.S. \tag{A.26}$$

References

- [1] B. E. W. Nilsson, C. N. Pope, *Hopf fibration of eleven-dimensional supergravity*, Class.Quant.Grav. **1**, 499 (1984).
- [2] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, *Superstrings In D = 10 From Supermembranes In D = 11*, Phys. Lett. B **191**, 70 (1987).

- [3] M. J. Duff, H. Lu and C. N. Pope, *Supersymmetry without supersymmetry*, Phys. Lett. B **409**, 136 (1997) [hep-th/9704186].
- [4] M. J. Duff, H. Lu and C. N. Pope, *AdS(5) x S(5) untwisted*, Nucl. Phys. B **532**, 181 (1998) [hep-th/9803061].
- [5] M. J. Duff, H. Lu and C. N. Pope, *AdS(3) x S**3 (un)twisted and squashed, and an O(2,2,Z) multiplet of dyonic strings*, Nucl. Phys. B **544**, 145 (1999) [hep-th/9807173].
- [6] I. R. Klebanov and A. A. Tseytlin, *Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories*, Nucl. Phys. B **578**, 123 (2000) [hep-th/0002159].
- [7] P. Candelas, X. C. de la Ossa, *Comments on Conifolds*, Nucl. Phys. **B342**, 246 (1990).
- [8] I. R. Klebanov and M. J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities*, JHEP **0008**, 052 (2000) [hep-th/0007191].
- [9] B. S. Acharya, *On realising N = 1 super Yang-Mills in M theory* [hep-th/0011089].
- [10] M. Atiyah, J. Maldacena and C. Vafa, *An M-theory flop as a large n duality* [hep-th/0011256].
- [11] C. Vafa, *Superstrings and topological strings at large N* [hep-th/0008142].
- [12] R. Gregory, J. A. Harvey and G. Moore, *Unwinding strings and T-duality of Kaluza-Klein and H-monopoles*, Adv. Theor. Math. Phys. **1**, 283 (1997) [hep-th/9708086].
- [13] R. Minasian and D. Tsimpis, *On the geometry of non-trivially embedded branes*, Nucl. Phys. B **572**, 499 (2000) [hep-th/9911042].
- [14] S. S. Gubser and I. R. Klebanov, *Baryons and domain walls in an N = 1 superconformal gauge theory*, Phys. Rev. D **58**, 125025 (1998) [hep-th/9808075].
- [15] K. Dasgupta and S. Mukhi, *Brane constructions, fractional branes and anti-de Sitter domain walls*, JHEP **9907**, 008 (1999) [hep-th/9904131].
- [16] I. R. Klebanov and N. A. Nekrasov, *Gravity duals of fractional branes and logarithmic RG flow*, Nucl. Phys. B **574**, 263 (2000) [hep-th/9911096].
- [17] M. J. Duff, B. E. W. Nilsson, C. N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. **130**, 1 (1986).
- [18] E. Witten, *Solutions of four-dimensional field theories via M-theory*, Nucl. Phys. B **500**, 3 (1997) [hep-th/9703166].
- [19] E. Witten, *Branes and the dynamics of QCD*, Nucl. Phys. B **507**, 658 (1997) [hep-th/9706109].
- [20] K. Dasgupta and S. Mukhi, *Brane constructions, conifolds and M-theory*, Nucl. Phys. B **551**, 204 (1999) [hep-th/9811139].
- [21] M. Cvetic, H. Lu and C. N. Pope, *Decoupling limit, lens spaces and Taub-NUT: D = 4 black hole microscopics from D = 5 black holes*, Nucl. Phys. B **549**, 194 (1999) [hep-th/9811107].

- [22] L. A. Pando Zayas and A. A. Tseytlin, *3-branes on resolved conifold*, JHEP **0011**, 028 (2000) [hep-th/0010088].
- [23] J. M. Maldacena and C. Nunez, *Towards the large N limit of pure $N = 1$ super Yang Mills*, Phys. Rev. Lett. **86**, 588 (2001) [hep-th/0008001].
- [24] J. D. Edelstein and C. Nunez, *D6 branes and M-theory geometrical transitions from gauged supergravity*, JHEP **0104**, 028 (2001) [hep-th/0103167].
- [25] K. Dasgupta, K. Oh and R. Tatar, *Geometric transition, large N dualities and MQCD dynamics* [hep-th/0105066].
- [26] M. Cvetič, H. Lu and C. N. Pope, *Massless 3-branes in M-theory* [hep-th/0105096].
- [27] M. Aganagic and C. Vafa, *Mirror symmetry and a $G(2)$ flop* [hep-th/0105225].
- [28] M. Cvetič, G. W. Gibbons, H. Lu and C. N. Pope, *M3-branes, $G(2)$ manifolds and pseudo-supersymmetry* [hep-th/0106026].
- [29] A. Brandhuber, J. Gomis, S. S. Gubser and S. Gukov, *Gauge theory at large N and new $G(2)$ holonomy metrics* [hep-th/0106034].